STRICTLY CONVEX NORMS AND TOPOLOGY

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Abstract. We introduce a new topological property called $(*)$ and the corresponding class of topological spaces, which includes spaces with G_δ-diagonals and Gruenhage spaces. Using $(*)$, we characterise those Banach spaces which admit equivalent strictly convex norms, and give an internal topological characterisation of those scattered compact spaces K, for which the dual Banach space $C(K)^*$ admits an equivalent strictly convex dual norm. We establish some relationships between $(*)$ and other topological concepts, and the position of several well-known examples in this context. For instance, we show that $C(\mathcal{K})^*$ admits an equivalent strictly convex dual norm, where $\mathcal{K}$ is Kunen's compact space. Also, under the continuum hypothesis CH, we give an example of a compact scattered non-Gruenhage space having $(*)$.

1. Introduction

Hereafter, all Banach spaces will be assumed real and, unless explicitly stated otherwise, all topological spaces will be Hausdorff. Throughout this paper we will be defining new norms on existing Banach spaces. These new norms will always be equivalent to the given canonical norms. Banach space notation and terminology is standard throughout.

A norm $\|\cdot\|$ on a Banach space X is said to be *strictly convex* (or rotund) if, given $x, y \in X$ satisfying $\|x\| = \|y\| = \|\frac{1}{2}(x+y)\|$, we have $x = y$ [5, p. 404]. Geometrically, this means that the unit sphere S_X of X in this norm has no non-trivial line segments, or, equivalently, every element of S_X is an extreme point of the unit ball B_X.

Clearly, there are many Banach spaces whose natural norms are not strictly convex. However, by appealing to the linear and topological properties of a given space, it is often possible to define a new norm that is strictly convex. Changing the norm in this way is often called *renorming*. In certain cases, we would like the new norm

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to possess in addition some form of lower semicontinuity. For instance, we may wish for a norm on a dual space X^* to be w^*-lower semicontinuous, so that it is the dual of some norm on X. Alternatively, we may like a norm on a $C(K)$-space to be lower semicontinuous with respect to the topology of pointwise convergence. Such additional requirements can make norms much more difficult to construct, but they do bestow certain benefits. For example, if X^* can be endowed with a strictly convex dual norm then the predual norm on X is automatically Gâteaux smooth, by virtue of Šmulyan's Lemma, cf. [8, Theorem I.1.4].

Despite the natural and intuitive nature of strict convexity, the question of whether a Banach space may be given such a norm turns out to be rather difficult to answer in general. A number of mathematicians have sought to establish more easily verifiable sufficient conditions and necessary conditions for a space to admit a strictly convex norm. Before outlining this paper, we mention some of the contributions to this collective endeavour. Specialists will realise that it is possible to endow many (but not all) of the spaces below with norms sporting stronger properties than strict convexity, but we prefer not to dwell on such properties here. For a fuller discussion, we refer the reader to [8, 36].

In [5, Theorem 9], it is shown that every separable Banach space admits a strictly convex norm. By following Clarkson's proof, Day showed that if a Banach space X is separable then X^* admits a strictly convex dual norm [6, Theorem 4]. If Γ is a set then $c_0(\Gamma)$ admits a strictly convex norm [6, Theorem 10] (see also [8, Definition II.7.2]). On the other hand, if Γ is uncountable then the space $\ell_\infty^c(\Gamma)$ of countably supported bounded functions $x : \Gamma \longrightarrow \mathbb{R}$ with the supremum norm is simply too big to admit a strictly convex norm [6, Theorem 8] ([8, Theorem II.7.12]).

Amir and Lindenstrauss showed that if X is weakly compactly generated (WCG) then both X and X^* admit a strictly convex norm and a strictly convex dual norm, respectively [2, Theorem 3]. These results rely on the fact that if a bounded linear map $T : X \longrightarrow Y$ is injective and Y admits a strictly convex norm, then so does X. If X is WCG then we can find such maps on both X and X^*, where $Y = c_0(\Gamma)$ for some Γ. Then [6, Theorem 10] can be applied.

At the time, such a 'linear transfer' into some $c_0(\Gamma)$ was the only way of showing that spaces admitted strictly convex norms. Moreover, $\ell_\infty^c(\Gamma)$, Γ uncountable, was the 'smallest' space known not to admit a strictly convex norm. In [9], the authors construct an increasing transfinite sequence $(X_\alpha)_{1\leq\alpha<\omega_1}$ of spaces of Baire-1 functions on $[0, 1]$, all admitting strictly convex norms, and none admitting a bounded linear injective map into any $c_0(\Gamma)$, provided $\alpha \geq 2$. Moreover, by refining Day's argument [6, Theorem 8], they showed that the union $Y = \bigcup_{\alpha<\omega_1} X_\alpha$ does not admit a strictly convex norm, and that there is no bounded linear injective map from $\ell_\infty^c([0, 1])$ into Y.

The fact that the dual of every WCG space admits strictly convex dual norm, with a necessarily Gâteaux smooth predual norm, prompted Lindenstrauss to conjecture that if X admits a Gâteaux smooth norm then it must embed as a subspace of some WCG space [21]. Mercourakis provided a negative answer to this conjecture by showing that if X is a *weakly countably determined* (WCD) space, then both X

and X^* admit strictly convex norms [22, Theorems 4.6 and 4.8], by virtue of linear transfers (although not into $c_0(\Gamma)$ in general).

Papers such as [9, 22] suggest that there is no simple way of characterising strict convexity in terms of linear structure. Since then, the problem of classifying Banach spaces admitting strictly convex norms has been approached from a more topological perspective, and particular attention has been paid to strictly convex dual norms and $C(K)$-spaces. Any Banach space X embeds isometrically into $C(B_{X^*}, w^*)$, and this fact enables certain results about $C(K)$-spaces to be generalised to all Banach spaces, by phrasing them in terms of the topological structure of (B_{X^*}, w^*).

For example, if X^* admits a strictly convex *dual* norm then (B_{X^*}, w^*) is fragmentable [30, Theorem 1.1]. We can say that a topological space is *fragmentable* if it admits, for each $n \in \mathbb{N}$, an increasing well ordered family of open subsets $(U_\xi)_{\xi<\lambda_n}$, with the property that given distinct points x and y, we can find some n_0 and $\xi < \lambda_{n_0}$ such that $\{x, y\} \cap U_\xi$ is a singleton [29, Theorem 1.9]. The idea of point separation features throughout this paper. Indeed, the notion of strict convexity can be viewed as a form of point separation.

The necessity condition above is far from sufficient however. The class of fragmentable spaces is very large and includes, for instance, all scattered spaces. Recall that a topological space is *scattered* if every non-empty subspace admits a relatively isolated point. In the year before [22] appeared, Talagrand showed that the space $C(\omega_1 + 1)^*$ does not admit a strictly convex dual norm [38, Théorème 3], where ω_1 is the first uncountable ordinal considered in its (scattered) order topology. On the other hand, the dual unit ball $(B_{C(\omega_1+1)^*}, w^*)$ is fragmentable [29, Theorem 3.1].

The next significant sufficiency condition we mention requires a definition.

Definition 1.1. A compact space K is *descriptive* if it admits a *σ-isolated network*, that is to say, a family $\mathscr{N} = \bigcup_{n=1}^\infty \mathscr{N}_n$ of subsets of K, satisfying

(1) $N \cap \overline{\bigcup \mathscr{N}_n \setminus \{N\}}$ is empty whenever $N \in \mathscr{N}_n$ and $n \in \mathbb{N}$, and
(2) if $x \in U \subseteq K$, where U is open, then there exists $n \in \mathbb{N}$ and $N \in \mathscr{N}_n$ such that $x \in N \subseteq U$.

This topological covering property arose out of the theory of 'generalised metric spaces' [12]. The class of descriptive compact spaces is large. For example, if X is WCD then (B_{X^*}, w^*) is descriptive ([37, Théorème 3.6] and [28, Corollary 2.4]). In [28, Theorem 3.3], Raja showed that if K is *descriptive* then $C(K)^*$ admits a strictly convex dual norm. This result can be adapted to give a sufficient condition which applies to a wide class of dual Banach spaces [26, Theorem 1.3], including duals of WCD spaces. We remark that a compact scattered space K is descriptive if and only if it is *σ-discrete*, that is, $K = \bigcup_{n=1} D_n$, where each D_n is discrete in its relative topology. This fact follows from [28, Lemma 2.2].

Despite these advances, there is a very large gap between the class of descriptive spaces and $\omega_1 + 1$ and the more general class of fragmentable spaces. Some years prior to the publication of [28], Haydon constructed some strictly convex dual norms on spaces of the form $C(K)^*$, where the K are 1-point compactifications of certain trees in their interval topologies [16, Theorem 7.1]. It turns out that some of these

spaces are not descriptive, so Haydon's sufficient condition is not covered by Raja's umbrella.

In [33, Theorem 6], the second-named author generalised Haydon's result by characterising those trees for which the associated spaces $C(K)^*$ admit strictly convex dual norms. Later, in [34], this order-theoretic characterisation was reproved in internal, topological terms. To state this result we need another definition.

Definition 1.2. A compact space K is called *Gruenhage* if there exists a sequence $(\mathscr{U}_n)_{n=1}^{\infty}$ of families of open subsets of K, and sets R_n, $n \geq 1$, with the property that

(1) if $x, y \in K$ are distinct, then there exists $n \in \mathbb{N}$ and $U \in \mathscr{U}_n$, such that $\{x, y\} \cap U$ is a singleton, and
(2) $U \cap V = R_n$ whenever $U, V \in \mathscr{U}_n$ are distinct.

This definition is equivalent to the original [13, p. 372] (see [34, Proposition 2]). Every descriptive compact space is Gruenhage [34, Corollary 4].

Theorem 1.3 ([34, Theorems 7 and 16])**.** *Let K be compact. Then the following statements hold.*

(1) *If K is Gruenhage then $C(K)^*$ admits a strictly convex dual lattice norm.*
(2) *If K is the 1-point compactification of a tree and $C(K)^*$ admits a strictly convex dual norm, then K is Gruenhage.*

Theorem 1.3 (1) can be adapted to give a sufficient condition ([34, Corollary 10]) which applies to class of dual Banach spaces even wider than that covered by [26, Theorem 1.3]. There are other instances of necessity besides Theorem 1.3 (2). For instance, if the Banach space X has an (uncountable) unconditional basis then X^* admits a strictly convex dual norm if and only if (B_{X^*}, w^*) is Gruenhage (equivalently, if (B_{X^*}, w^*) is descriptive) [35, Theorem 6]. Despite some courageous attempts, it was not possible to prove the converse implication of Theorem 1.3 (1). Many of the results of this paper are the product of efforts to resolve this difficulty.

This paper is organised as follows. In Section 2, we introduce a generalisation of Gruenhage's property, labeled $(*)$ (Definition 2.6), and use it to give a characterisation of Banach spaces which admit a strictly convex norm satisfying some additional lower semicontinuity property (Theorem 2.8). This characterisation attempts to topologise as much as possible the geometric condition of strict convexity. In Section 3, we use $(*)$ to find an analogue of Theorem 1.3 which applies to all scattered compact spaces (Theorem 3.1). This class is significant in Banach space theory because $C(K)$ is an *Asplund* space if and only if K is scattered. In doing so, we show that $(*)$ comes close to providing a complete topological characterisation of those K, for which $C(K)^*$ admits a strictly convex dual norm. In Section 4, we establish some of the topological properties of $(*)$ and its position in the wider context of covering properties, and provide some examples of scattered compact spaces, some of which having $(*)$ and others not. In particular, we give an example of a scattered non-Gruenhage compact space having $(*)$ (Example 4.10). Thus, Theorem 3.1 does not follow from previous results such as Theorem 1.3. Along the way, we answer an open question concerning Kunen's compact space: specifically, we show that it is Gruenhage (Proposition 4.7). In several cases, including Example 4.10, we

shall assume extra principles independent of the usual axioms of set theory. Finally, in Section 5, we present some open problems stemming from this study.

2. A characterisation of strict convexity in Banach spaces

In this section, we provide a general characterisation of strictly convex renormings in Banach spaces. Throughout this section, X will be a Banach space (and occasionally a general topological space) and $F \subseteq X^*$ a norming subspace. Recall that $\sigma(X, F)$ denotes the coarsest topology on X with respect to which every element of F is continuous. We begin by presenting a useful folklore result, together with a brief sketch proof.

Proposition 2.1. *Let $F \subseteq X^*$ be a norming subspace. Suppose that there exists a sequence of $\sigma(X, F)$-lower semicontinuous convex functions $\varphi_n : X \longrightarrow [0, \infty)$ such that given distinct $x, y \in X$, we can find $n \in \mathbb{N}$ satisfying*

$$\varphi_n(\tfrac{1}{2}(x + y)) < \max\{\varphi(x), \varphi(y)\}. \tag{1}$$

Then X admits a $\sigma(X, F)$-lower semicontinuous strictly convex norm $|||\cdot|||$. Instead, if X is a Banach lattice, (1) holds whenever $x, y \in X_+$ are distinct, and

$$\varphi_n(x) \leq \varphi_n(y)$$

whenever $|x| \leq |y|$ and $n \in \mathbb{N}$, then $|||\cdot|||$ is a $\sigma(X, F)$-lower semicontinuous strictly convex lattice norm.

Proof. Let $\|\cdot\|$ denote the original norm on X. We define a new norm by

$$|||x|||^2 = \sum_{n,q} c_{n,q} \|x\|_{n,q}^2$$

where $\|\cdot\|_{n,q}$ is the Minkowski functional of

$$C_{n,q} = \left\{ x \in X \ : \ \varphi_n(x)^2 + \varphi_n(-x)^2 \leq q \right\}$$

whenever q is a rational number satisfying $q > 2\varphi_n(0)^2$, and where the constants $c_{n,q} > 0$ are chosen to ensure the uniform convergence of the sum on bounded sets. By a standard convexity argument (cf. [8, Fact II.2.3]), it can be shown that if $|||x||| = |||y||| = \frac{1}{2}|||x + y|||$ then $\varphi_n(x) = \varphi_n(y) = \varphi_n(\frac{1}{2}(x + y))$ for all n, whence $x = y$ by hypothesis. If we adopt the lattice hypotheses instead then clearly $|||\cdot|||$ is also a lattice norm, and strictly convex on X_+. To see that the strict convexity extends to all of X, let $x, y \in X$ and suppose that $|||x||| = |||y||| = \frac{1}{2}|||x + y|||$. Then $\frac{1}{2}|||\,|x| + |y|\,||| = |||x|||$ as well, so strict convexity on X_+ yields $|x| = |y|$. If we set $w = \frac{1}{2}(x + y)$ then repeating the above gives us $|x| = |w|$. A simple lattice argument (e.g. [34, p. 749]) leads us to conclude that $x = y$. □

Our characterisation adopts several ideas from [25, 24]. Recall that if A is a subset of a locally convex space then an open *slice* U of A is the intersection of A with an open half-space of X. The following proposition will be our main tool.

Proposition 2.2. *Let A be a bounded subset of X and $\mathscr{U}$ a family of non-empty $\sigma(X,F)$-open slices of A. Then there exists a $\sigma(X,F)$-lower semicontinuous 1-Lipschitz convex function φ with the property that whenever $x, y \in A$, $\{x,y\} \cap \bigcup \mathscr{U}$ is non-empty and*

$$\varphi(x) \;=\; \varphi(y) \;=\; \varphi(\tfrac{1}{2}(x+y)),$$

we have $x, y \in U$ for some $U \in \mathscr{U}$.

Proposition 2.2 is an immediate corollary of the next result, dubbed the 'Slice Localisation Theorem'.

Theorem 2.3 ([25, Theorem 3])**.** *Let A be a bounded subset of X and $\mathscr{U}$ a family of non-empty $\sigma(X,F)$-open slices of A. Then there is an equivalent $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|$ such that for every sequence $(x_n)_{n=1}^{\infty} \subseteq X$ and $x \in A \cap \bigcup \mathscr{U}$, if*

$$2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \to 0,$$

then there is a sequence of slices $(U_n)_{n=1}^{\infty} \subseteq \mathscr{U}$ and $n_0 \in \mathbb{N}$ such that

1. *$x, x_n \in U_n$ whenever $n \geq n_0$ and $x_n \in A$;*
2. *for every $\delta > 0$ there is some $n_\delta \in \mathbb{N}$ such that*
$$x, x_n \in \overline{(\mathrm{conv}(A \cap U_n) + \delta B_X)}^{\sigma(X,F)}$$
for all $n \geq n_\delta$.

The Slice Localisation Theorem can be used to simplify the proofs of network characterisations of Banach spaces which admit locally uniformly rotund norms. To prove Proposition 2.2, all we need to do apply Theorem 2.3 with $x_n = y$ for all n. However, there is a more transparent proof of this proposition which we provide for completeness.

Of key importance to the proof is the concept of F-distance, introduced in [24]. Let $D \subseteq X$ be a non-empty, convex bounded subset. Given $\xi \in X^{**}$, define

$$\|\xi\|_F \;=\; \sup\{\xi(f) \;:\; f \in B_{X^*} \cap F\}. \tag{1}$$

It is clear that $\|\cdot\|_F$ is $\sigma(X^{**},F)$-lower semicontinuous ($\sigma(X^{**},F)$ being the only generally non-Hausdorff topology mentioned in this paper). Now set

$$\varphi(x) \;=\; \inf\left\{\|x-d\|_F \;:\; d \in \overline{D}^{\sigma(X^{**},X^*)}\right\}.$$

Definition 2.4. Given a non-empty, convex bounded subset $D \subseteq X$, we call $\varphi(x)$ the *F-distance* from $x \in X$ to D.

We pass to the bidual of X in order to control the lower semicontinuity properties of φ. The notion of F-distance has a number of useful properties which we list in the next lemma.

Lemma 2.5. *Let $\varphi(x)$ be the F-*distance *from $x \in X$ to D.*

1. *φ is convex and 1-Lipschitz;*
2. *φ is $\sigma(X,F)$-lower semicontinuous;*
3. *$\overline{D}^{\sigma(X,F)} = \varphi^{-1}(0)$.*

Properties (1) and (2) are proved in [24, Proposition 2.1] and the third is a straightforward exercise involving the Hahn-Banach separation theorem. Now we can give our alternative proof of Proposition 2.2.

Proof of Proposition 2.2. For each $U \in \mathscr{U}$ and $x \in X$, define $\varphi_U(x)$ to be the F-distance from x to $(\operatorname{conv} A) \setminus U$. Since A is bounded, we can define another convex, $\sigma(X, F)$-lower semicontinuous, 1-Lipschitz function by

$$\varphi(x) \;=\; \sup\{\varphi_U(x) \;:\; U \in \mathscr{U}\}.$$

Let $x, y \in A$ with $\{x, y\} \cap \bigcup \mathscr{U}$ non-empty and suppose that

$$\varphi(x) = \varphi(y) = \varphi(\tfrac{1}{2}(x + y)).$$

Without loss of generality, we can assume that $x \in U$ for some $U \in \mathscr{U}$. Since $U \cap \overline{(\operatorname{conv} A) \setminus U}^{\sigma(X,F)}$ is empty, we have $\varphi(x) \geq \varphi_U(x) > 0$ by Lemma 2.5, part (3). Pick $\varepsilon > 0$ such that $\varphi(x) > 5\varepsilon^2$ and choose $V \in \mathscr{U}$ with the property that

$$\varphi(\tfrac{1}{2}(x + y))^2 < \varphi_V(\tfrac{1}{2}(x + y))^2 + \varepsilon^2.$$

We have

$$\begin{aligned} 0 &= \tfrac{1}{2}(\varphi(x)^2 + \varphi(y)^2) - \varphi(\tfrac{1}{2}(x + y))^2 \\ &> \tfrac{1}{2}(\varphi_V(x)^2 + \varphi_V(y)^2) - \varphi_V(\tfrac{1}{2}(x + y))^2 - \varepsilon^2 \\ &\geq \tfrac{1}{2}(\varphi_V(x)^2 + \varphi_V(y)^2) - \tfrac{1}{4}(\varphi_V(x) + \varphi_V(y))^2 - \varepsilon^2 \\ &= \tfrac{1}{4}(\varphi_V(x) - \varphi_V(y))^2 - \varepsilon^2 \end{aligned}$$

thus

$$|\varphi_V(x) - \varphi_V(y)| \;<\; 2\varepsilon. \tag{2}$$

Since φ_V is convex, we have $\max\{\varphi_V(x), \varphi_V(y)\} \geq \varphi_V(\frac{1}{2}(x+y))$. Together with (2), this implies

$$\begin{aligned} \min\{\varphi_V(x), \varphi_V(y)\} &\geq \max\{\varphi_V(x), \varphi_V(y)\} - 2\varepsilon \\ &\geq \varphi_V(\tfrac{1}{2}(x + y)) - 2\varepsilon \\ &\geq \left(\varphi(\tfrac{1}{2}(x + y)) - \varepsilon^2\right)^{\frac{1}{2}} - 2\varepsilon \\ &> 0. \end{aligned}$$

Therefore $\varphi_V(x), \varphi_V(y) > 0$. Since $x, y \in A$, we get $x, y \in V$. $\square$

Proposition 2.2 motivates the introduction of the central topological concept featuring in this paper.

Definition 2.6. We say that a topological space X has $(*)$ if there exists a sequence $(\mathscr{U}_n)_{n=1}^{\infty}$ of families of open subsets of X, with the property that given any $x, y \in X$, there exists $n \in \mathbb{N}$ such that

(1) $\{x, y\} \cap \bigcup \mathscr{U}_n$ is non-empty, and
(2) $\{x, y\} \cap U$ is at most a singleton for all $U \in \mathscr{U}_n$.

Any sequence $(\mathscr{U}_n)_{n=1}^\infty$ satisfying the conditions of Definition 2.6 will be called a *$(*)$-sequence* for X. In addition, if X is locally convex and $A \subseteq X$ then we say A has *$(*)$ with slices* if A admits a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$, with the property that every element of $\bigcup_{n=1}^\infty \mathscr{U}_n$ is an open slice of A.

Remark 2.7. It will be convenient to note that if $A \subseteq X$, then to say that $(A, \sigma(X,F))$ has $(*)$ with slices is equivalent to there being a family of subsets $G_n \subseteq (S_{X^*} \cap F) \times \mathbb{R}$, $n \in \mathbb{N}$, such that given distinct $x, y \in A$, we have $n \in \mathbb{N}$ satisfying

(a) $\max\{f(x), f(y)\} > \lambda$ for some $(f,\lambda) \in G_n$, and
(b) $\min\{g(x), g(y)\} \leq \mu$ for every $(g,\mu) \in G_n$.

Our characterisation follows.

Theorem 2.8. *Let $F \subseteq X^*$ be a 1-norming subspace. Then the following are equivalent.*

(1) *X admits a $\sigma(X,F)$-lower semicontinuous strictly convex norm;*
(2) *$(X, \sigma(X,F))$ has $(*)$ with slices;*
(3) *$(S_X, \sigma(X,F))$ has $(*)$ with slices;*
(4) *there is a sequence of subsets $(X_n)_{n=1}^\infty$ of X, such that*

$$\{(x,y) \in X^2 \ : \ x \neq y\} \subseteq \bigcup_{n=1}^\infty X_n^2$$

and where each $(X_n, \sigma(X,F))$ has $(*)$ with slices.

Proof. (1) $\Rightarrow$ (2): let $\|\cdot\|$ be a $\sigma(X,F)$-lower semicontinuous strictly convex norm on X. Then F is also 1-norming for $\|\cdot\|$. Let

$$G_q = (S_{(X,\|\cdot\|)^*} \cap F) \times \{q\}$$

for each rational number $q > 0$. We verify that $(X, \sigma(X,F))$ has $(*)$ by showing that the G_q satisfy (a) and (b) of Remark 2.7. Given distinct $x, y \in X$, assume that $\|x\| \leq \|y\|$. The strict convexity of $\|\cdot\|$ tells us that $\|\frac{1}{2}(x+y)\| < \|y\|$. Let rational q satisfy $\|\frac{1}{2}(x+y)\| < q < \|y\|$. Since F is 1-norming for $\|\cdot\|$, we know that $f(y) > q$ for a pair $(f,q) \in G_q$, giving (a). Now suppose $g(y) > q$ for some $(g,q) \in G_q$. Then certainly $g(x) \leq q$, else we would have

$$q < \tfrac{1}{2}g(x+y) \leq \tfrac{1}{2}\|x+y\|,$$

which doesn't make any sense. This shows that (b) is also satisfied.

(2) $\Rightarrow$ (3) is trivial because $(*)$ with slices is inherited by subspaces. (3) $\Rightarrow$ (2): if $(S_X, \sigma(X,F))$ has $(*)$ with slices then we take sets G_n, $n \in \mathbb{N}$ that satisfy (a) and (b) of Remark 2.7. We can assume that $G_n \subseteq (S_{X^*} \cap F) \times (-1,1)$ for every n. Given rational $q, r > 0$, set

$$H_q = (S_{X^*} \cap F) \times \{q\} \qquad \text{and} \qquad L_{n,q,r} = \{(f, q(\lambda + r)) \ : \ (f,\lambda) \in G_n\}.$$

We claim that the H_q and $L_{n,q,r}$ verify that $(X, \sigma(X,F))$ has $(*)$, using Remark 2.7.

Let $x, y \in X$ be distinct, with $\|x\| \leq \|y\|$. If $\|x\| < \|y\|$ then we choose rational q to satisfy $\|x\| < q < \|y\|$. Since F is 1-norming, it is easy to check that (a) and (b)

are fulfilled by H_q. Now suppose $\|x\| = \|y\|$. We know that, with respect to $x/\|x\|$ and $y/\|y\|$, (a) and (b) are satisfied by some G_n. Without loss of generality, assume $f(x) > \|x\|\lambda$, where $(f, \lambda) \in G_n$. Our argument depends on the sign of λ. If $\lambda \geq 0$ then choose rational $q, r > 0$ satisfying

$$f(x) > \|x\|(\lambda + r) \qquad \text{and} \qquad \frac{\|x\|}{1+r} < q < \|x\|.$$

The constants have been arranged to ensure

$$\mu(\|x\| - q) < \|x\| - q < qr \qquad \text{whenever } |\mu| < 1. \tag{3}$$

We have $f(x) > \|x\|(\lambda + r) > q(\lambda + r)$. Now suppose that $g(x) > q(\mu + r)$, where $(g, \mu) \in G_n$. Then

$$g(x) > q(\mu + r) > \|x\|\mu$$

by equation (3) above. This means $g(x/\|x\|) > \mu$, whence $g(y/\|y\|) \leq \mu$ by (b), giving $g(y) < q(\mu + r)$. In summary, we have shown that (a) and (b) of Remark 2.7 are fulfilled by $L_{n,q,r}$. If instead $\lambda < 0$, we choose $r < -\lambda$ as above and ensure that q satisfies

$$\|x\| < q < \frac{\|x\|}{1-r}.$$

By arguing similarly, we get what we want.

(2) $\Rightarrow$ (4) follows easily by setting $X_n = X$. We finish by proving (4) $\Rightarrow$ (1). By taking intersections with mB_X, $m \in \mathbb{N}$, and reindexing if necessary, we can assume that each X_n is bounded. Let each X_n have a $(*)$-sequence $(\mathscr{U}_{n,m})_{m=1}^{\infty}$, where each element of $\bigcup_{m=1}^{\infty} \mathscr{U}_{n,m}$ is a (non-empty) $\sigma(X, F)$-open slice of X_n. Let $\varphi_{n,m}$ denote the convex function constructed by applying Proposition 2.2 to X_n and the family $\mathscr{U}_{n,m}$. We have ensured that if $x, y \in X$ are distinct then we can find n and m such that $\varphi_{n,m}(\frac{1}{2}(x + y)) < \max\{\varphi_{n,m}(x), \varphi_{n,m}(y)\}$. The rest follows from Proposition 2.1. $\square$

Note that Theorem 2.8 (1), (2) and (4) are also equivalent when F is simply a norming subspace, rather than a 1-norming subspace. We end this section by giving an example to show that the reliance on slices in the statement of Theorem 2.8 is necessary in general.

Example 2.9. Let K be the product $\{0, 1\}^{\omega_1}$, endowed with the lexicographic order topology. According to [17, Example 1], $C(K)$ admits a Kadec norm $\|\cdot\|$ but no strictly convex norm. By the definition of Kadec norms, the weak topology agrees with the norm topology on $S_{(C(K),\|\cdot\|)}$. In particular $(S_{(C(K),\|\cdot\|)}, w)$ is metrisable, meaning that it has a σ-discrete base and thus has $(*)$ as well. However, since $\|\cdot\|$ cannot be strictly convex, Theorem 2.8 implies that $(S_{(C(K),\|\cdot\|)}, w)$ does not have $(*)$ *with slices*.

We conclude this section by giving a sufficient condition for constructing strictly convex norms. Theorem 2.11 below can be applied to many spaces of significance to the theory, such as the Mercourakis spaces $c_1(\Sigma' \times \Gamma)$ (see [8, Section VI.6]), Dashiell-Lindenstrauss spaces and spaces of the form $C(K)^*$, where K is Gruenhage. The idea, which goes back to the classical norm of Day for $c_0(\Gamma)$ [6, Theorem 10], is to

'glue together' strictly convex norms on finite-dimensional spaces (which are readily available) to obtain strictly convex norms on larger spaces. Elements of Theorem 2.11 can be found in [11, Theorem 5]. Before giving the theorem, we state a simple fact.

Fact 2.10. *Let $\xi : [0,1] \longrightarrow \mathbb{R}$ be a function satisfying $\xi(0)\xi(1) < 0$, and suppose that ξ_+ and ξ_- are convex. Then for every $\lambda \in (0,1)$, we have*

$$\xi_\pm(\lambda) < \lambda\xi_\pm(1) + (1-\lambda)\xi_\pm(0). \tag{4}$$

Proof. Assume $\xi(0) > 0$. Since $\xi_\pm$ are convex and ξ is necessarily continuous, it is easy to see that there is a unique interval $[a,b]$, where $0 < a \leq b < 1$, such that

$$\xi(u) > \xi(v) = 0 > \xi(w)$$

whenever $u \in [0,a)$, $v \in [a,b]$ and $w \in (b,1]$. If $\lambda \in [a,b]$ then clearly equation (4) holds for $\xi_\pm$. Let $\lambda < a$. Then $\xi_-(\lambda) = 0$ and, as $\xi_-(1) > 0$, (4) holds for ξ_-. Since ξ_+ is convex, setting $\mu = \lambda/a$ gives

$$\xi_+(\lambda) \leq (1-\mu)\xi_+(0) + \mu\xi_+(a) = (1-\mu)\xi_+(0) < (1-\lambda)\xi_+(0)$$

so (4) holds for ξ_+. The proof for the case $\lambda > b$ is similar. $\square$

Clearly, if ξ is linear then $\xi_\pm$ are convex. The same is true if ξ is positive and convex.

Theorem 2.11. *Let $\Theta_n : X \longrightarrow \ell_\infty(\Gamma_n)$ be a sequence of maps such that both functions $x \mapsto \Theta_{n,\pm}(x)(\gamma)$ are $\sigma(X,F)$-lower semicontinuous and convex for every $\gamma \in \Gamma_n$ and $n \in \mathbb{N}$.*

Let us assume in addition that for all distinct $x, y \in X$, there are $\lambda \in (0,1)$, $n \in \mathbb{N}$ and a finite set $A \subseteq \Gamma_n$, such that

$$\Theta_n(x)\restriction_A \neq \Theta_n(y)\restriction_A, \quad \textit{and} \tag{1}$$

$$|\Theta_n(z)(\alpha)| > |\Theta_n(z)(\gamma)| \quad \textit{whenever } \alpha \in A \textit{ and } \gamma \in \Gamma \setminus A, \tag{2}$$

where $z = \lambda x + (1-\lambda)y$. Then X admits a $\sigma(X,F)$-lower semicontinuous strictly convex norm $|||\cdot|||$.

Instead, if X is a Banach lattice, $\Theta_{n,\pm}(x) \leq \Theta_{n,\pm}(y)$ whenever $|x| \leq |y|$ and equations (1) and (2) apply to distinct $x, y \in X_+$, then $|||\cdot|||$ is a $\sigma(X,F)$-lower semicontinuous strictly convex lattice norm.

Proof. Since $\Theta_{n,\pm}(\cdot)(\gamma)$ are both convex and $\sigma(X,F)$-lower semicontinuous, the same is true of $|\Theta_n(\cdot)(\gamma)|$. Define

$$\Theta_{n,0}(x)(\gamma) = \Theta_n^2(x)(\gamma) \qquad \text{and} \qquad \Theta_{n,\pm1}(x)(\gamma) = \Theta_{n,\pm}(x)(\gamma).$$

If $\Gamma = \bigcup_{n=1}^\infty \Gamma_n$, $u \in \ell_\infty(\Gamma)$ and $A \subseteq \Gamma$ is finite, set

$$\varphi_A(u) = \sum_{\gamma\in A} u(\gamma)$$

and put $\varphi_{A,n,i} = \varphi_A \circ \Theta_{n,i}$ for every $n \in \mathbb{N}$ and $i \in \{-1,0,1\}$. Certainly, each $\varphi_{A,n,i}$ is $\sigma(X,F)$-lower semicontinuous, non-negative and convex. Finally, let

$$\psi_{m,n,i}(x) = \sup\{\varphi_{A,n,i}(x) \; : \; A \subseteq \Gamma_n \text{ has cardinality } m\}.$$

To finish the proof, we shall show that for every distinct pair $x, y \in X$, there is $m, n \in \mathbb{N}$ and $i \in \{-1, 0, 1\}$ such that

$$\psi_{m,n,i}(\tfrac{1}{2}(x+y)) < \max\{\psi_{m,n,i}(x),\, \psi_{m,n,i}(y)\}. \tag{5}$$

holds. Then we can appeal to Proposition 2.1.

Take $\lambda \in (0,1)$, $n \in \mathbb{N}$ and $A \subseteq \Gamma_n$ satisfying (1) and (2). We consider two cases. First suppose that $\Theta_n(x)(\beta)\Theta_n(y)(\beta) < 0$ for some $\beta \in A$. From (2) we know that $\Theta_n(z)(\beta) \neq 0$. Assume for now that $\Theta_n(z)(\beta) > 0$ and define the non-empty set

$$B = \{\alpha \in A \ : \ \Theta_{n,+}(z)(\alpha) > 0\}.$$

so that

$$\Theta_{n,+}(z)(\alpha) > \Theta_{n,+}(z)(\gamma)$$

for every $\alpha \in B$ and $\gamma \in \Gamma \setminus B$. Therefore $\psi_{n,m,1}(z) = \sum_{\alpha \in B} \Theta_n(z)(\alpha)$, where m is the cardinality of B. Applying Fact 2.10 to $\xi(t) = \Theta_n(tx + (1-t)y)(\beta)$, $t \in [0,1]$, we get

$$\Theta_{n,+}(z)(\beta) < \lambda\Theta_{n,+}(x)(\beta) + (1-\lambda)\Theta_{n,+}(y)(\beta)$$

whence

$$\psi_{n,m,1}(z) < \lambda\psi_{n,m,1}(x) + (1-\lambda)\psi_{n,m,1}(y)$$

from which (5) quickly follows for $i = 1$, by convexity. If $\Theta_n(z)(\beta) < 0$ then we argue similarly with $i = -1$.

Let's now consider the case

$$\Theta_n(x)(\alpha)\Theta_n(y)(\alpha) \geq 0 \tag{6}$$

for all $\alpha \in A$. Let $m \in \mathbb{N}$ be the cardinality of A. Since $t \mapsto t^2$ is strictly convex, from condition (1) we have

$$\begin{aligned}
\sum_{\alpha \in A}(\lambda\Theta_n(x)(\alpha) + (1-\lambda)\Theta_n(y)(\alpha))^2 &< \sum_{\alpha \in A}\lambda(\Theta_n(x)(\alpha))^2 + (1-\lambda)(\Theta_n(y)(\alpha))^2 \\
&= \lambda\varphi_{A,n,0}(x) + (1-\lambda)\varphi_{A,n,0}(y) \\
&\leq \lambda\psi_{m,n,0}(x) + (1-\lambda)\psi_{m,n,0}(y) \\
&\leq \max\{\psi_{m,n,0}(x),\, \psi_{m,n,0}(y)\}.
\end{aligned}$$

Given the convexity of $|\Theta_n(\cdot)(\alpha)|$ and equation (6), we obtain

$$|\Theta_n(z)(\alpha)| = |\Theta_n(\lambda x + (1-\lambda)y)(\alpha)| \leq |\lambda\Theta_n(x)(\alpha) + (1-\lambda)\Theta_n(y)(\alpha)|.$$

This and condition (2) imply

$$\psi_{m,n,0}(z) \ = \ \varphi_{A,n,0}(z) \ \leq \ \sum_{\alpha \in A}(\lambda\Theta_n(x)(\alpha) + (1-\lambda)\Theta_n(y)(\alpha))^2.$$

Combining these inequalities we see that

$$\psi_{m,n,0}(z) < \max\{\psi_{m,n,0}(x), \psi_{m,n,0}(y)\}$$

from which (5) follows for $i = 0$, again by convexity. If we adopt the lattice assumptions instead, then each $\psi_{n,m,i}$ satisfies the lattice assumptions in Proposition 2.1. □

In the first corollary below is a sufficient condition of 'Mercourakis type', which is formally more general than similar conditions given in the literature.

Corollary 2.12. *Let X be a subspace or sublattice of $\ell_\infty(\Gamma)$ and suppose that there are subsets $\Gamma_n \subseteq \Gamma$, $n \in \mathbb{N}$, with the property that given $x \in X$ and $\alpha \in \operatorname{supp} x$, we can find n and $\alpha \in \Gamma_n$, so that*

$$\{\gamma \in \Gamma_n : |x(\gamma)| \geq |x(\alpha)|\}$$

is finite. Then X admits a pointwise lower semicontinuous strictly convex norm or lattice norm, respectively.

Proof. Let $P_n(x)(\gamma) = |x(\gamma)|$ whenever $\gamma \in \Gamma_n$ and $n \in \mathbb{N}$. The coordinate maps are positive and convex. We show that P_n satisfies conditions (1) and (2) of Theorem 2.11. Given distinct $x, y \in X$, take $n \in \mathbb{N}$ and $\beta \in \Gamma_n$ such that $x(\beta) \neq y(\beta)$. Then there is $\lambda \in (0,1)$ such that $\lambda x(\beta) + (1-\lambda)y(\beta)$ is non-zero. Set $z = \lambda x + (1-\lambda)y$ and take $n \in \mathbb{N}$ such that

$$A = \{\alpha \in \Gamma_n \ : \ |z(\alpha)| \geq |z(\beta)|\}$$

is finite. Evidently $\beta \in A$, so $P_n(x)\restriction_A \neq P_n(y)\restriction_A$, and

$$|P_n(z)(\alpha)| \geq |z(\beta)| > |P_n(z)(\gamma)|$$

whenever $\alpha \in A$ and $\gamma \in \Gamma_n \setminus A$. □

Corollary 2.13 ([34, Theorem 7])**.** *If K is Gruenhage then $C(K)^*$ admits a strictly convex dual lattice norm.*

Proof. If K is Gruenhage then (cf. [34, Lemma 6]), we can find sequences $(\mathscr{U}_n)_{n=1}^\infty$ and $(R_n)_{n=1}^\infty$ as in Definition 1.2, with the further property that if $\mu \in C(K)^*$ and $\mu(U) = 0$ for all $U \in \mathscr{U}_n$, $n \in \mathbb{N}$, then $\mu = 0$. Let $\Gamma_n = \mathscr{U}_n$ and define

$$\Theta_n(\mu)(U) = |\mu|(U), \qquad U \in \mathscr{U}_n.$$

Since $|\lambda\mu + (1-\lambda)\nu| \leq \lambda|\mu| + (1-\lambda)|\nu|$ whenever $\lambda \in [0,1]$, the coordinate maps $\Theta_n(\cdot)(U)$ are positive and convex. If $\mu, \nu \in C(K)^*$ are positive and distinct, then there exists $n \in \mathbb{N}$ and $U \in \mathscr{U}_n$ such that $\mu(U) \neq \nu(U)$. If we set $\tau = \frac{1}{2}(\mu + \nu)$ then we have $\tau(U) > \tau(R_n)$. By considering Definition 1.2 part (2), we see that for any $r > \tau(R_n)$, there are only finitely many $V \in \mathscr{U}_n$ satisfying $\tau(V) \geq r$. Therefore, conditions (1) and (2) of Theorem 2.11 apply to positive elements of $C(K)^*$. Now we are able to apply Theorem 2.11. □

Dashiell-Lindenstrauss spaces can be shown to have strictly convex lattice norms in a similar way.

3. Strictly convex dual norms on $C(K)^*$

Evidently, Theorem 2.8 relies on geometric assumptions, in the sense that only sets having $(*)$ with *slices* are considered. According to Example 2.9, it is not always possible to remove the reliance on slices and deal instead with open sets having no special geometric properties. However, we can live without slices in an important special case. We devote this section to proving the next result.

Theorem 3.1. *Let K be a scattered compact space. Then $C(K)^*$ admits a strictly convex dual (lattice) norm if and only if K has $(*)$.*

Recall that any compact space K embeds naturally into $(C(K)^*, w^*)$ by identifying points $t \in K$ with their Dirac measures δ_t. It follows therefore from Theorem 3.1 that if K is scattered and $(C(K)^*, w^*)$ has $(*)$ (without slices), then $(C(K)^*, w^*)$ has $(*)$ with slices. One implication of Theorem 3.1 may be proved easily.

Proposition 3.2. *If $C(K)^*$ admits a strictly convex dual norm then K has $(*)$.*

Proof. By Theorem 2.8, if $C(K)^*$ admits a strictly convex dual norm then $(C(K)^*, w^*)$ has $(*)$, whence K has $(*)$ by the natural embedding. □

In order to prove the converse implication, we need to refine our $(*)$-sequences so that they satisfy some additional properties. Assume that a topological space X admits a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^{\infty}$. Given any finite sequence of natural numbers $\sigma = (n_1, \dots, n_k)$, we define the family

$$\mathscr{U}_\sigma = \left\{ \bigcap_{i=1}^{k} U_i \ : \ U_i \in \mathscr{U}_{n_i} \text{ for all } i \leq k \right\}.$$

Let us also set $C_n = \bigcup \mathscr{U}_n$ and $C_\sigma = \bigcup \mathscr{U}_\sigma$.

Lemma 3.3. *Assume that $F \subseteq X$ is a finite subset such that for all n, either $F \cap C_n = \varnothing$ or $F \subseteq C_n$. Then there exists $\sigma = (n_1, \dots, n_k)$ such that $F \subseteq C_\sigma$ and, moreover, $F \cap V$ is at most a singleton for all $V \in \mathscr{U}_\sigma$.*

Proof. Enumerate the set of doubletons $\{x, y\} \subseteq F$ as $\{x_1, y_1\}, \dots, \{x_k, y_k\}$. For every i, there exists n_i such that $\{x_i, y_i\} \cap C_{n_i}$ is non-empty and $\{x_i, y_i\} \cap V$ is at most a singleton for all $V \in \mathscr{U}_{n_i}$. By hypothesis, we have $F \subseteq C_{n_i}$ for all i. Put $\sigma = (n_1, \dots, n_k)$. If $x \in F$, since $F \subseteq C_{n_i}$ for all i, let $U_i \in \mathscr{U}_{n_i}$ so that $x \in \bigcap_{i=1}^k U_i \in \mathscr{U}_\sigma$. Therefore $F \subseteq C_\sigma$. Given $V = \bigcap_{i=1}^k V_i \in \mathscr{U}_\sigma$ and distinct $x, y \in F$, we have some i such that $\{x, y\} \cap W$ is at most a singleton for all $W \in \mathscr{U}_{n_i}$. In particular, $\{x, y\} \cap V \subseteq \{x, y\} \cap V_i$ is at most a singleton. This proves that $F \cap V$ is at most a singleton for any $V \in \mathscr{U}_\sigma$. □

Bearing in mind the $\mathscr{U}_\sigma$, Lemma 3.3, and by adding new singleton families if necessary, if X has $(*)$ then we can assume that there exists a $(*)$-sequence with additional properties, which we list in the next lemma.

Lemma 3.4. *If X has $(*)$ then it admits a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^{\infty}$ with the following properties.*

1. *$X = C_1$;*
2. *given $n_1, \dots, n_k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that*
$$\mathscr{U}_m = \left\{ \bigcap_{i=1}^{k} U_i \ : \ U_i \in \mathscr{U}_{n_i} \textit{ for all } i \leq k \right\};$$
3. *if F is a finite subset of X such that for each $n \in \mathbb{N}$, either $F \subseteq C_n$ or $F \cap C_n$ is empty, then there exists $m \in \mathbb{N}$ with two properties:*

(a) $F \subseteq C_m$;
(b) $F \cap V$ *is at most a singleton for all* $V \in \mathscr{U}_m$.

Armed with these enhanced $(*)$-sequences, we can deliver the proof of Theorem 3.1. We ask that our compact spaces be scattered because the proof relies on the assumption that all measures in $C(K)^*$ are atomic.

Proof of Theorem 3.1. One implication was proved in Proposition 3.2. Now assume that K is scattered and let $(\mathscr{U}_n)_{n=1}^{\infty}$ be a $(*)$-sequence for K satisfying the properties of Lemma 3.4. Given $n \geq 1$, $k \geq 0$ and finite $L \subseteq \mathbb{N}$, define the seminorm

$$\|\mu\|_{n,k,L} = \sup \left\{ |\mu| \left(\bigcup_{i \in L} C_i \cup \bigcup \mathscr{F} \right) \;:\; \mathscr{F} \subseteq \mathscr{U}_n \text{ and } \operatorname{card} \mathscr{F} = k \right\}.$$

We show that these seminorms satisfy the requirements of Proposition 2.1. To this end, suppose that μ and ν are positive, and that

$$\|\mu\|_{n,k,L} = \|\nu\|_{n,k,L} = \tfrac{1}{2}\|\mu + \nu\|_{n,k,L} \tag{7}$$

for all n, k and L. For a contradiction, we shall suppose also that $\mu \neq \nu$. Since

$$\|\mu\|_{1,0,\{n\}} = \mu(C_n)$$

we have $\mu(C_n) = \nu(C_n) = \frac{1}{2}(\mu + \nu)(C_n)$ for all n, by (7). By Lemma 3.4 (1) and (2), and the inclusion-exclusion principle, if $I \subseteq \mathbb{N}$ then we know that

$$\mu(C_{I,n}) = \nu(C_{I,n}) = \tfrac{1}{2}(\mu + \nu)(C_{I,n})$$

where

$$C_{I,n} \quad = \quad \bigcap_{i \leq n, i \in I} C_i \setminus \bigcup_{i \leq n, i \notin I} C_i.$$

By monotone convergence, it follows that

$$\mu(C_I) = \nu(C_I) = \tfrac{1}{2}(\mu + \nu)(C_I)$$

where

$$C_I = \bigcap_{i \in I} C_i \setminus \bigcup_{i \in \mathbb{N} \setminus I} C_i.$$

Now K is the disjoint union of the C_I, where I ranges over non-empty subsets of $\mathbb{N}$, and since $\mu \neq \nu$ are atomic, we can find non-empty $I \subseteq \mathbb{N}$ such that $\mu\restriction_I \neq \nu\restriction_I$. We fix this I from now on. Take a countable set $A \subseteq C_I$ such that we can write

$$\mu\restriction_I = \sum_{t \in A} a_t \delta_t \qquad \text{and} \qquad \nu\restriction_I = \sum_{t \in A} b_t \delta_t$$

for some numbers a_t, $b_t \geq 0$. Let

$$p = \max \{\max\{a_t, b_t\} \;:\; t \in A, a_t \neq b_t\}$$

$$q = \max(\{a_t \;:\; a_t < p\} \cup \{b_t \;:\; b_t < p\})$$

and define the finite, possibly empty, set

$$F = \{t \in A \;:\; a_t = b_t \geq p\}$$

and let $k = \operatorname{card} F$. Take finite $G \subseteq A$ such that

$$(8) \qquad \sum_{t \in A \setminus G} a_t, \ \sum_{t \in A \setminus G} b_t < \tfrac{1}{4}(p-q)$$

and n large enough so that

$$(9) \qquad \mu(C_{I,n} \setminus C_I), \, \nu(C_{I,n} \setminus C_I) < \tfrac{1}{4}(p-q).$$

Let $H = \{1, \ldots, n\} \cap I$ and $L = \{1, \ldots, n\} \setminus I$. By Lemma 3.4 (3), we can find $m \in \mathbb{N}$ such that $G \subseteq C_m$ and $G \cap V$ is at most a singleton for all $V \in \mathscr{U}_m$. Since $C_I \subseteq \bigcap_{i \in H} C_i$, we can and do assume that $C_m \subseteq \bigcap_{i \in H} C_i$, by Lemma 3.4 (2).

It is by considering the seminorm $\| \cdot \|_{m,k+1,L}$ that we reach our contradiction. Let $u \in A$ such that $a_u \neq b_u$ and $\max\{a_u, b_u\} = p$. Clearly $u \notin F$. Also, $F \cup \{u\} \subseteq G$. Indeed, if $t \in A \setminus G$ then $a_t, b_t < \frac{1}{4}(p-q) < p$. Without loss of generality, assume that $a_u < b_u = p$. Since $F \cup \{u\} \subseteq C_m$, it is possible to find $\mathscr{G} \subseteq \mathscr{U}_m$ of cardinality $k+1$, such that $F \cup \{u\} \subseteq \bigcup \mathscr{G}$.

By considering $\| \cdot \|_{1,0,L}$ and (7), we know that $\mu\left(\bigcup_{i \in L} C_i\right) = \nu\left(\bigcup_{i \in L} C_i\right)$. We shall denote this common quantity by c. We estimate

$$(10) \qquad \begin{aligned} \|\nu\|_{m,k+1,L} &\geq \nu\left(\bigcup_{i \in L} C_i \cup \bigcup \mathscr{G}\right) \\ &\geq \nu\left(\bigcup_{i \in L} C_i\right) + \sum_{t \in F \cup \{u\}} b_t \qquad \text{as } (F \cup \{u\}) \cap \bigcup_{i \in L} C_i = \varnothing \\ &\geq c + p + \sum_{t \in F} b_t \ = \ c + p + \sum_{t \in F} a_t. \end{aligned}$$

By (7) and the definition of the seminorms, let $\mathscr{H} \subseteq \mathscr{U}_m$ of cardinality $k+1$ be chosen in such a way that

$$\tfrac{1}{2}(\mu + \nu)\left(\bigcup_{i \in L} C_i \cup \bigcup \mathscr{H}\right) > \|\nu\|_{m,k+1,L} - \tfrac{1}{4}(p-q).$$

We claim that $a_t \geq p$ whenever $t \in \bigcup \mathscr{H} \cap G$. In order to see this, first of all we claim that if $J \subseteq A$ has cardinality at most k, then

$$(11) \qquad \sum_{t \in J} a_t \leq \sum_{t \in F} a_t.$$

Indeed, we have $\operatorname{card} F \setminus J \geq \operatorname{card} J \setminus F$, since $\operatorname{card} J \leq k = \operatorname{card} F$. If $t \in J \setminus F$ then either $a_t < p$ or $a_t \neq b_t$, which means $a_t \leq p$ by maximality of p. Therefore

$$\begin{aligned} \sum_{t \in F} a_t - \sum_{t \in J} a_t &= \sum_{t \in F \setminus J} a_t - \sum_{t \in J \setminus F} a_t \\ &\geq p(\operatorname{card} F \setminus J) - p(\operatorname{card} J \setminus F) \ \geq \ 0. \end{aligned}$$

This finishes the proof of the claim.

Now we can show that $a_t \geq p$ whenever $t \in \bigcup \mathscr{H} \cap G$. If not, then $a_s < p$ for some $s \in \bigcup \mathscr{H} \cap G$, meaning $a_s \leq q$. Observe that

$$\bigcup_{i \in L} C_i \cup \bigcup \mathscr{H} \subseteq \left(\bigcup \mathscr{H} \cap G\right) \cup \left(\bigcup \mathscr{H} \cap C_I \setminus G\right) \cup (C_{I,n} \setminus C_I) \cup \bigcup_{i \in L} C_i. \tag{12}$$

To see this, it helps to note that

$$\bigcup \mathscr{H} \setminus \bigcup_{i \in L} C_i \subseteq C_m \setminus \bigcup_{i \in L} C_i \subseteq \bigcap_{i \in H} C_i \setminus \bigcup_{i \in L} C_i = C_{I,n}.$$

By choice of m, $\operatorname{card} \bigcup \mathscr{H} \cap G \leq k+1$. Hence

$$\begin{aligned} \mu\left(\bigcup_{i \in L} C_i \cup \bigcup \mathscr{H}\right) &\leq \sum_{t \in F} a_t + a_s + \tfrac{1}{4}(p-q) + \tfrac{1}{4}(p-q) + c \quad \text{by (8), (9), (11), (12)} \\ &\leq \sum_{t \in F} a_t + q + \tfrac{1}{2}(p-q) + c \quad \text{since } a_s \leq q \\ &\leq \|\nu\|_{m,k+1,L} - \tfrac{1}{2}(p-q) \qquad \text{by (10).} \end{aligned}$$

However, this means

$$\tfrac{1}{2}(\mu+\nu)\left(\bigcup_{i \in L} C_i \cup \bigcup \mathscr{H}\right) \leq \tfrac{1}{2}\|\nu\|_{m,k+1,L} - \tfrac{1}{4}(p-q) + \tfrac{1}{2}\|\nu\|_{m,k+1,L}$$

which contradicts the choice of $\mathscr{H}$. Therefore $a_t \geq p$ whenever $t \in \bigcup \mathscr{H} \cap G$. By a similar argument applied to the b_t, we have $b_t \geq p$ whenever $t \in \bigcup \mathscr{H} \cap G$. Hence we know that $a_t = b_t$ for $t \in \bigcup \mathscr{H} \cap G$, lest we contradict the maximality of p. It follows that $\bigcup \mathscr{H} \cap G \subseteq F$. However, this forces

$$\begin{aligned} \mu\left(\bigcup_{i \in L} C_i \cup \bigcup \mathscr{H}\right) &\leq \sum_{t \in F} a_t + \tfrac{1}{4}(p-q) + \tfrac{1}{4}(p-q) + c \qquad \text{by (8), (9) and (12)} \\ &< \|\nu\|_{m,k+1,L} - \tfrac{1}{2}(p-q). \end{aligned}$$

Just as above, this contradicts the choice of $\mathscr{H}$. $\square$

Remark 3.5. Most of Theorem 3.1 follows from Theorem 2.8. Starting with a $(*)$-sequence from Lemma 3.4, we can show directly that $(C(K)^*, w^*)$ has $(*)$ with slices. For $n, k \in \mathbb{N}$, finite $L \subseteq \mathbb{N}$ and rational $q > 0$, define $\mathscr{V}_{n,k,L,q,+}$ to be the family of all w^*-open sets

$$\left\{\mu \in C(K)^* \ :\ \mu_+\left(\bigcup_{i \in L} C_i \cup \bigcup \mathscr{F}\right) > q\right\}$$

where $\mathscr{F} \subseteq \mathscr{U}_n$ has cardinality k. Define $\mathscr{V}_{n,k,L,q,-}$ accordingly. By using essentially the same method as that presented above, it can be shown that the $\mathscr{V}_{n,k,L,q,\pm}$ form a $(*)$-sequence. Moreover, if $V \in \mathscr{V}_{n,k,L,q,\pm}$ then $C(K)^* \setminus V$ is convex. By the Hahn-Banach Theorem, each such V can be written as a union of w^*-open half-spaces. Therefore, we can write down a $(*)$-sequence for $(C(K)^*, w^*)$, the elements of which being families of half-spaces. What we lose here is the fact that the norm in Theorem 3.1 is a lattice norm, which is why we give the proof as is.

4. Topological properties of $(*)$ and examples

In this section, we explore the properties of $(*)$ and see how it compares with related concepts in the literature. In particular, under the continuum hypothesis (CH), we provide an example of a compact scattered non-Gruenhage space K having $(*)$. This means that Theorem 3.1 does not follow from existing results such as Theorem 1.3.

A topological space X is said to have a G_δ*-diagonal* if its diagonal

$$\{(x,x) \ : \ x \in X\}$$

is a G_δ set in X^2. This concept has been studied extensively in general metrisation theory; see, for example [12, Section 2]. It is easy to show that X has a G_δ-diagonal if and only if the admits a sequence $(\mathscr{G}_n)_{n=1}^\infty$ of open covers of X, such that given $x, y \in X$, there exists n with the property that $\{x,y\} \cap U$ is at most a singleton for all $U \in \mathscr{G}_n$ [12, Theorem 2.2]. Equivalently, if we consider the 'stars'

$$\mathrm{st}(x,n) = \bigcup \{U \in \mathscr{G}_n \ : \ x \in U\},$$

then $\bigcap_{n=1}^\infty \mathrm{st}(x,n) = \{x\}$ for every $x \in X$. In keeping with previous notation, we call such a sequence a G_δ*-diagonal sequence*. Compact spaces with G_δ-diagonals are metrisable (cf. [12, Theorem 2.13]), so $(*)$ is evidently a strict generalisation of the G_δ-diagonal property. In some cases, it is possible to reduce problems about $(*)$ to the G_δ-diagonal case; see Theorem 4.3 and Proposition 4.12, and also the partitioning of K into the C_I in the proof of Theorem 3.1.

Next, we compare $(*)$ with Gruenhage's property.

Proposition 4.1. *If X is Gruenhage then it has $(*)$.*

Proof. If X is Gruenhage then let $(\mathscr{U}_n)_{n=1}^\infty$ and R_n be as in Definition 1.2. Let $\mathscr{V}_n = \{R_n\}$ for each n. Given distinct $x, y \in X$, there exists n and $U \in \mathscr{U}_n$, such that $\{x,y\} \cap U$ is a singleton. If $x \in R_n$ then $y \notin R_n$ and it is true that $\{x,y\} \cap U = \{x\}$ for every $U \in \mathscr{V}_n$, because $\mathscr{V}_n$ is a singleton. Likewise if $y \in R_n$. So we assume now that $x, y \notin R_n$. Now it is true that $\{x,y\} \cap V$ is at most a singleton for every $V \in \mathscr{U}_n$, since if $y \in V$ then $V \neq U$, and if $x \in V$ then $x \in U \cap V = R_n$. $\square$

There are an abundance of compact spaces which are Gruenhage, but non-descriptive and so quite far from being metrisable; see [34, Corollary 17] or Theorem 4.6 and subsequent remarks, below. In Example 4.10, we show that under CH there exists a compact, scattered non-Gruenhage space that has $(*)$. Now we see that $(*)$ implies fragmentability.

Proposition 4.2. *If X has $(*)$ then X is fragmentable.*

Proof. Let X have a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$. We well order each $\mathscr{U}_n$ as $(U^n_\xi)_{\xi<\lambda_n}$. Now define $V^n_\alpha = \bigcup_{\xi\leq\alpha} U^n_\xi$, for $\alpha < \lambda_n$. We claim that given distinct $x, y \in X$, there exists n and $\alpha < \lambda_n$ such that $\{x,y\} \cap V^n_\alpha$ is a singleton. As explained in the Introduction, this is enough to give fragmentability. Indeed, take $n \in \mathbb{N}$ with the

properties given in Definition 2.6, and pick the least $\alpha < \lambda_n$ such that $\{x,y\} \cap U_\alpha^n$ is a singleton. Then $\{x,y\} \cap U_\xi^n$ must be empty for all $\xi < \alpha$, thus

$$\{x,y\} \cap V_\alpha^n \;=\; \{x,y\} \cap U_\alpha^n$$

is a singleton. □

Theorem 4.3 below is a generalisation of a result of Chaber (cf. [12, Theorem 2.14]), which states that countably compact spaces with G_δ-diagonals are compact (and thus metrisable). It allows us to glean a few more topological consequences of the $(*)$ property. As preparation, fix an open cover $\mathscr{V}$ of a countably compact (non-empty) space X. Suppose that X has a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$, with $C_n = \bigcup \mathscr{U}_n$ for each n. Define

$$\mathscr{A}_X = \left\{ I \subseteq \mathbb{N} \;:\; X \setminus \bigcup_{n\in I} C_n \neq \varnothing \right\}.$$

Clearly, $\mathscr{A}_X$ is a hereditary family of subsets of $\mathbb{N}$. Moreover, it is compact in the pointwise topology. Indeed, if $J \notin \mathscr{A}_X$, then by the countable compactness of X, we can find finite $G \subseteq J$ such that $G \notin \mathscr{A}_X$. It follows that $\mathbb{P}(\mathbb{N}) \setminus \mathscr{A}_X$ is open. Furthermore, $\varnothing \in \mathscr{A}_X$ because X is non-empty, so $\mathscr{A}_X$ is also non-empty. From these facts, we deduce that $\mathscr{A}_X$ admits an element that is maximal with respect to inclusion.

Theorem 4.3. *If X is countably compact and has $(*)$ then X is compact.*

Proof. Fix an open cover $\mathscr{V}$ of X and $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$ as above. We define a decreasing transfinite sequence of countably compact subspaces X_α of X, together with maximal $M_\alpha \in \mathscr{A}_{X_\alpha}$ and finite $\mathscr{F}_\alpha \subseteq \mathscr{V}$, such that

(1) $X_\alpha = X \setminus \bigcup_{\xi<\alpha} \bigcup \mathscr{F}_\xi$;
(2) $M_\xi \notin \mathscr{A}_{X_\alpha}$ whenever $\xi < \alpha$.

To begin, set $X_0 = X$. Given X_α, we take some maximal $M_\alpha \in \mathscr{A}_{X_\alpha}$ and set $Y = X_\alpha \setminus \bigcup_{n\in M_\alpha} C_n$. We claim that $(\mathscr{U}_n)_{n\in\mathbb{N}\setminus M_\alpha}$ is a G_δ-diagonal sequence for Y. Indeed, the maximality of M_α implies that $Y \subseteq C_n$ whenever $n \in \mathbb{N}\setminus M_\alpha$. If $x,y \in Y$ then by $(*)$, there exists n such that $\{x,y\} \cap C_n$ is non-empty, and $\{x,y\} \cap U$ is at most a singleton for all $U \in \mathscr{U}_n$. By definition, $Y \cap C_k$ is empty whenever $k \in M_\alpha$, so necessarily $n \in \mathbb{N} \setminus M_\alpha$. Our claim is proved.

By Chaber's result, Y is compact. Therefore there exists a finite set $\mathscr{F}_\alpha \subseteq \mathscr{V}$, such that

$$X_\alpha \setminus \bigcup_{n\in M_\alpha} C_n = Y \subseteq \bigcup \mathscr{F}_\alpha.$$

Define $X_{\alpha+1} = X_\alpha' = X_\alpha \setminus \bigcup \mathscr{F}_\alpha$. We have (1) immediately and (2) follows because $M_\alpha \notin \mathscr{A}_{X_{\alpha+1}}$ and $\mathscr{A}_{X_{\alpha+1}} \subseteq \mathscr{A}_{X_\alpha}$. If $X_{\alpha+1}$ is empty then we stop the recursion. If λ is a countable limit ordinal and X_α is non-empty for all $\alpha < \lambda$, set $X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha$. (1) and (2) follow. By countable compactness, X_λ is also non-empty.

This process has to stop at a countable (successor) stage, because $(\mathscr{A}_{X_\alpha})$ is a strictly decreasing family of closed subsets of the separable metric space $\mathbb{P}(\mathbb{N})$. Thus,

$X_{\alpha+1}$ is empty for some $\alpha < \omega_1$. By (1), we get

$$X \subseteq \bigcup_{\xi \leq \alpha} \bigcup \mathscr{F}_\xi$$

and so X is covered by $\bigcup_{\xi \leq \alpha} \mathscr{F}_\xi$. By a final application of countable compactness, we extract from this a finite subcover. $\square$

The next result generalises [26, Corollary 4.3] from descriptive spaces to spaces with $(*)$.

Corollary 4.4. *If L is locally compact and has $(*)$ then $L \cup \{\infty\}$ is countably tight and sequentially closed subsets of $L \cup \{\infty\}$ are closed.*

Proof. The first assertion follows directly from Theorem 4.3 and the second follows from Proposition 4.2 and the fact that compact fragmentable spaces are sequentially compact (see [29, Corollary 2.7] and [10, Lemma 2.1.1]). Notice that if L is any locally compact space with $(*)$ then its 1-point compactification $L \cup \{\infty\}$ has $(*)$ also. All we need to do is adjoin to any $(*)$-sequence for L the singleton family $\{L\}$, which separates all points in L from ∞. $\square$

Concerning stability properties of $(*)$ under mappings, we have the next result.

Proposition 4.5. *If K is a scattered compact space with $(*)$ and $\pi : K \longrightarrow M$ is a continuous, surjective map then M has $(*)$.*

Proof. If K has $(*)$ then by Theorem 3.1, $C(K)^*$ admits a strictly convex dual norm $\|\cdot\|$. If we define $T : C(M) \longrightarrow C(K)$ by $T(f) = f \circ \pi$, it is standard to check that

$$|||\nu||| = \inf \{\|\mu\| \ : \ T^*(\mu) = \nu\}$$

defines a strictly convex dual norm on $C(M)^*$. Therefore M has $(*)$, again by Theorem 3.1. $\square$

The proof above is concise and straightforward, but also utterly opaque, as it leaves the reader with no idea of how to construct a $(*)$-sequence on M in terms of a $(*)$-sequence on K. We outline a second approach to proving Proposition 4.5, which we include because we believe it gives the reader more idea of what is going on. The dual map $S = T^*$ above is a natural extension of π if we identify points in K and M with their Dirac measures in $C(K)^*$ and $C(M)^*$, respectively. Set

$$\Sigma = \{\mu \in C(K)^* \ : \ \mu \text{ is positive and } \|\mu\|_1 = 1\}.$$

If $t \in M$ and $\mu \in \Sigma$ then $S(\mu) = t$ if and only if $\operatorname{supp} \mu \subseteq \pi^{-1}(t)$. Given a $(*)$-sequence $(\mathscr{U}_n)_{n=1}^\infty$ on K with the properties of Lemma 3.4, together with the unions C_n, define the w^*-compact and convex sets

$$D_{n,q,L} = \left\{ \mu \in \Sigma \ : \ \mu \left(\bigcup_{i \in L} C_i \cup U \right) \leq q \text{ for all } U \in \mathscr{U}_n \right\}$$

where $n \in \mathbb{N}$, $q \in (0,1) \cap \mathbb{Q}$ and $L \subseteq \mathbb{N}$ is finite. The $D_{n,q,L}$ should be compared to the seminorms $\|\cdot\|_{n,k,L}$ in the proof of Theorem 3.1. Given distinct $s, t \in M$ and $\mu, \nu \in \Sigma$ in $S^{-1}(s)$ and $S^{-1}(t)$ respectively, by following the proof of Theorem 3.1,

we can find n and q and L such that $\frac{1}{2}(\mu+\nu) \in D_{n,q,L}$, but $\{\mu,\nu\} \cap D_{n,q,L}$ is at most a singleton. There is less to consider in this case because as the supports of μ and ν are necessarily disjoint, the set F in the proof of Theorem 3.1 is empty. This is why we only need to consider individual elements of $\mathscr{U}_n$ in the definition of the $D_{n,q,L}$, rather than finite subsets of $\mathscr{U}_n$ as in the definition of the $\|\cdot\|_{n,k,L}$.

By appealing to compactness and convexity, it is possible to select a finite set G of triples (n,q,L) with the property that if we consider the intersection $D_G = \bigcup_{(n,q,L)\in G} D_{n,q,L}$, then $D_G \cap S^{-1}(\frac{1}{2}(s+t))$ is non-empty, but either $D_G \cap S^{-1}(s)$ is empty, or $D_G \cap S^{-1}(t)$ is empty. Equivalently, $\frac{1}{2}(s+t) \in S(D_G)$, but $\{s,t\} \cap S(D_G)$ is at most a singleton. The set $S(D_G)$ is w^*-compact and convex, so the complement $C(M)^* \setminus S(D_G)$ can be written as the union of a family $\mathscr{V}_G$ of w^*-open halfspaces of $C(M)^*$. From what we know, it can be easily verified that the families $\mathscr{V}_G$, as G ranges over all finite subsets of triples (n,q,L), induce a $(*)$-sequence on M.

Now we move on to examples. We are chiefly interested in exploring $(*)$, Gruenhage's property and the gap between them. Given that descriptive spaces are Gruenhage and spaces with $(*)$ are fragmentable, we shall confine our attention to spaces that are fragmentable but non-descriptive.

The first thing to point out is that $(*)$ is not equivalent to fragmentability, because ω_1 is scattered (hence fragmentable), but does not have $(*)$. That ω_1 does not have $(*)$ is clear, either directly from Corollary 4.4, or from Theorem 3.1 and [38, Théorème 3], which we mentioned in the Introduction. Any locally compact space having $(*)$ necessarily has a countably tight 1-point compactification, but this condition is not sufficient. Hereafter, all of our examples of locally compact spaces without $(*)$ have countably tight 1-point compactifications.

Next, we consider trees. A *tree* $(T,\leq)$ is a partially ordered set with the property that given any $t \in T$, its set of predecessors $\{s \in T : s \leq t\}$ is well ordered. The tree order induces a natural locally compact, scattered *interval topology.* To render this topology Hausdorff, we shall only consider trees T with the property that every non-empty totally ordered subset of T has at most one minimal upper bound. An antichain is a subset of T, no two distinct elements of which are comparable. For further definitions and discussions about trees, and their role in renorming theory, we refer the reader to [15, 16, 33, 34, 36, 39].

If P and Q are partially ordered sets then we say that a map $\rho : P \longrightarrow Q$ is *strictly increasing* if $\rho(x) < \rho(y)$ whenever $x < y$. If such a map exists then we write $P \preccurlyeq Q$. In [33, Definition 5], the second-named author introduced a totally ordered set Y to address the problem of when $C_0(T)^*$ admits a strictly convex dual norm. We remark of Y that $\mathbb{R} \preccurlyeq Y$, $Y^\alpha \preccurlyeq Y$ for all $\alpha < \omega_1$, where Y^α is ordered lexicographically, and finally Y contains no uncountable, well ordered subsets [33, Section 4]. By combining Theorem 3.1 with [34, Corollary 17], we obtain the next result. See also [36, Theorem 26].

Theorem 4.6. *If T is a tree then the following are equivalent.*

(1) *T is Gruenhage;*
(2) *T has $(*)$;*
(3) *$C_0(T)^*$ admits a strictly convex dual norm;*

(4) $T \preccurlyeq Y$.

Note that the 1-point compactification $T \cup \{\infty\}$ of a tree T is countably tight if and only if T admits no uncountable branches. Indeed, suppose that T admits no uncountable branches. Since each $t \in T$ admits a countable neighbourhood, the only point we need to test is ∞. If $\infty \in \overline{A}$ for some uncountable $A \subseteq T$, then by a standard result of Ramsey theory, either A contains an uncountable totally ordered set or a countably infinite antichain E. Only the second possibility is valid, whence $\infty \in \overline{E}$. The converse implication follows immediately from the fact that $\omega_1 + 1$ is not countably tight. Thus we restrict our attention to trees with no uncountable branches.

Given a partially ordered set P, we set

$$\sigma P = \{A \subseteq P \;:\; A \text{ is well-ordered}\}.$$

Kurepa introduced this notion and proved the following fact: for all P, we have $\sigma P \not\preccurlyeq P$. On the other hand, it is straightforward to show that $\sigma\mathbb{R}^\alpha \preccurlyeq \mathbb{R}^\alpha \times \{0,1\}$ [33, Proposition 23]. Moreover, it is known that T is descriptive if and only if $T \preccurlyeq \mathbb{Q}$ [33, Theorem 4]. Therefore, we conclude that $\sigma\mathbb{Q}$ and $\sigma\mathbb{R}^\alpha$, $\alpha < \omega_1$, are all Gruenhage, non-descriptive spaces (see [34, p. 752] or [36, p. 405]). Instead, if we consider any total order W satisfying $Y \preccurlyeq W$, then $\sigma W \not\preccurlyeq Y$ and so σW does not have $(*)$. In addition, if W doesn't contain any uncountable well ordered subsets, then σW is free of uncountable branches.

There is another type of tree without uncountable branches and without $(*)$. A subset E of a tree is a *final part* if $u \in E$ whenever $t \in E$ and $t \leq u$. If E is a final part then we say that E is *dense* if every element of T is comparable with some element of E, and T is called *Baire* if every countable intersection of dense final parts (which is itself a final part) is again dense. A subset E is called *ever-branching* if, given any $t \in E$, there exist incomparable elements $u, v \in E$ satisfying $t < u, v$. If T admits an ever-branching Baire subtree then $C_0(T)$ does not admit a Gâteaux norm [15, Theorem 2.1]. Therefore, no such tree can have $(*)$. An ever-branching Baire tree without uncountable branches exists; see [39, Lemma 9.12] and [15, Proposition 3.1]. Recall that a tree T is called *Suslin* if it contains no uncountable branches or antichains. The existence of Suslin trees is independent of ZFC; see, for example, [39, Section 6]. Every Suslin tree contains an ever-branching Baire subtree [39, p. 246], so we conclude that no Suslin tree has $(*)$ either.

It is clear from Theorem 4.6 that in order to find examples of non-Gruenhage spaces with $(*)$, we must search further afield. A topological space X is said to be *hereditarily separable* (HS) if every subspace of X is separable. Clearly, the 1-point compactification of a locally compact HS space is countably tight. These spaces are interesting for us because if K is compact, HS and non-metrisable, then it is automatically non-descriptive. This fact is stated in [26, Proposition 4.2] but no direct proof is given, so an argument is sketched here for completeness. If $\mathscr{H}$ is an isolated family of subsets of K then $\mathscr{H}$ must be countable, because by hereditary separability there is a countable subset of $\bigcup \mathscr{H}$ which meets every member of $\mathscr{H}$.

Therefore, if K is a descriptive compact HS space then it admits a countable network, whence it is metrisable.

Since we want compact, non-metrisable HS spaces that are also fragmentable, it is necessary to assume extra axioms. A space X is *hereditarily Lindelöf* (HL) if every subspace of X is Lindelöf. If K is compact, fragmentable and HL then it is metrisable (cf. [20, Corollary 9]). Thus, we want HS spaces that are not HL; such objects are called *S-spaces.* We refer the reader to [31] for an introduction to S-spaces and also the related L-spaces. It is known that under MA + $\neg$CH (where MA stands for Martin's axiom), there are no compact S-spaces (cf. [31, Theorem 6.4.1]), and in fact it is consistent that there are no S-spaces at all (cf. [31, Theorem 7.2.1]). Therefore, we must assume extra axioms if we are to find any animals in this particular zoo.

Our treatment of S-spaces proceeds as follows. First, we outline two approaches for constructing S-spaces by refining existing topologies, and show that these yield Gruenhage spaces. Second, we give an example under CH of a compact non-Gruenhage space of cardinality $\aleph_1$ with $(*)$ and show that, given a further mild assumption, no object of this kind can exist under MA + $\neg$CH. Finally, we present a third method of constructing S-spaces and show that no such space can have $(*)$.

The spaces developed using the first approach are sometimes called 'Kunen lines', despite the fact that none of them are linearly ordered. Assuming CH, the authors of [18] develop a machine which accepts as input a first countable HS space (X, ρ) of cardinality $\aleph_1$, and generates a finer topology (X, τ) which is locally compact, scattered, HS and non-Lindelöf. In applications, X is usually a subset of $\mathbb{R}$ and ρ is the induced metric topology.

Later, this process was developed to ensure that $(X, \tau)^n$ is HS for all $n \in \mathbb{N}$; [23, Section 7]. The resulting 1-point compactification $\mathcal{K}$ is known to Banach space theorists as 'Kunen's compact space'. It is not explicitly stated in [23, Section 7] that the resulting topology on X refines that of the real line, but the authors believe that it is meant to. If the topology is such a refinement then necessarily the Euclidean diameters of the B_k^α (which form the building blocks of neighbourhoods of points, see (8) on [23, p. 1124]) have to tend to 0 as $k \to \infty$. It can be checked that this condition is also sufficient to produce a refinement. We note further that an alternative approach to [23, Section 7] is given in [7, Theorem 2.4], and there, the fact that the original topology is refined is explicitly stated.

Of course, it is clear that any refinement of a Gruenhage space is again Gruenhage, because we can use exactly the same open sets to separate points. Therefore, assuming the adjustment to the diameters of the B_k^α above, we have the following result.

Proposition 4.7. *The Kunen lines are Gruenhage spaces. In particular, $C(\mathcal{K})^*$ admits a strictly convex dual norm, the predual of which is necessarily Gâteaux smooth.*

The second approach refines topologies as above, but this time using the axiom $\mathfrak{b} = \aleph_1$, where $\mathfrak{b}$ is the minimal cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which is unbounded with respect to the ordering of eventual dominance. Under $\mathfrak{b} = \aleph_1$, it is shown in

[40, Theorem 2.5] that the topology of any set of reals of cardinality $\aleph_1$ may be refined to give a locally compact, scattered, non-Lindelöf topology which is HS in its finite powers.

Proposition 4.8. *The spaces of Todorčević in* [40, Theorem 2.5] *are Gruenhage.*

Before presenting our third approach to construct S-spaces, we give our example under CH of a compact, scattered non-Gruenhage space with $(*)$. We shall adopt the same basic approach as [18] and [23, Section 7], and use an idea from [1]. However, the underlying motivation for the space should be compared, at a distance, to the split interval, rather than the real line.

In fact, we construct a locally compact, scattered non-Gruenhage space with a G_δ-diagonal. The 1-point compactification of this space has $(*)$. For our example, we shall make use of the following observation about Gruenhage spaces of cardinality no larger than the continuum.

Proposition 4.9. [36, Proposition 2] *Let X be a topological space with* $\operatorname{card} X \leq \mathfrak{c}$. *Then X is Gruenhage if and only if there is a sequence $(U_n)_{n=1}^\infty$ of open subsets of X with the property that if $x, y \in X$, then $\{x, y\} \cap U_n$ is a singleton for some n.*

In that which follows, diam denotes Euclidean diameter.

Example 4.10. (CH) There exists a locally compact, scattered, first countable Hausdorff, non-Gruenhage space with a G_δ-diagonal.

Proof. Let $(x_\alpha)_{\alpha<\omega_1}$ be a set of distinct points in $[0,1]$. Define $Y_\alpha = \{x_\xi \ : \ \xi < \alpha\}$ and $X_\alpha = Y_\alpha \times \{\pm 1\}$ for $\alpha \leq \omega_1$, with $Y = Y_{\omega_1}$ and $X = X_{\omega_1}$. Assuming CH, let $(A_\alpha)_{\alpha<\omega_1}$ be an enumeration of all the countable subsets of Y. Let $t : X \longrightarrow X$ be the map $t(x,i) = (x,-i)$, and let $q : X \longrightarrow Y$ be the natural projection. We obtain our topology on X by building increasing topologies τ_α on the X_α, $\alpha < \omega_1$, by transfinite induction. The points (x_α, i), $i = \pm 1$, will have a countable base of compact open neighbourhoods $U(x_\alpha, i, n)$, $n \in \mathbb{N}$, such that

(1) if $\beta < \alpha$ then X_β is open in τ_α and τ_β is the topology on X_β induced by τ_α;
(2) $U(x_\alpha, i, n) \setminus \{(x_\alpha, i)\} \subseteq X_\alpha$;
(3) $\operatorname{diam}(q(U(x_\alpha, i, n))) < 2^{-n}$;
(4) $U(x_\alpha, -i, n) = t(U(x_\alpha, i, n))$;
(5) $q\restriction_{U(x_\alpha,i,n)}$ is injective;
(6) if $\xi \leq \alpha$, $A_\xi \subseteq Y_\alpha$ and $x_\alpha \in \overline{A_\xi}^{\mathbb{R}}$, then
$$U(x_\alpha, i, n) \cap (A_\xi \times \{-i\})$$
is non-empty for every n.

To take care of limit stages α, we set
$$\tau_\alpha = \{U \subseteq X_\alpha \ : \ U \cap X_\beta \in \tau_\beta \text{ for all } \beta < \alpha\}.$$
Now assume that τ_α has been found. We construct $\tau_{\alpha+1}$ by constructing neighbourhoods $U(x_\alpha, i, n)$, $n \in \mathbb{N}$, of the points (x_α, i), $i = \pm 1$.

If $x_\alpha \notin \overline{Y_\alpha}^{\mathbb{R}}$ then set $U(x_\alpha, i, n) = \{(x_\alpha, i)\}$ for $i = \pm 1$ and $n \in \mathbb{N}$. Note that as Y is a separable subset of $\mathbb{R}$, this can happen for only countably many α. Now assume that $x_\alpha \in \overline{Y_\alpha}^{\mathbb{R}}$. Define

$$F_\alpha = \left\{ \xi \leq \alpha \ : \ A_\xi \subseteq Y_\alpha \text{ and } x_\alpha \in \overline{A_\xi}^{\mathbb{R}} \right\}.$$

Since F_α is at most countable, we can find an injective sequence $(s_n)_{n=1}^\infty \subseteq Y_\alpha$ converging to x_α, such that

(i) $\operatorname{diam}(\{s_m \ : \ m \geq n\}) < 2^{-n}$ for each n

(ii) $\{n \in \mathbb{N} \ : \ s_n \in A_\xi\}$ is infinite whenever $\xi \in F_\alpha$.

By considering (3) applied to $\beta < \alpha$, and (i) above, for every n we can find k_n such that

(iii) $q(U(s_n, -1, k_n)) \cap q(U(s_m, -1, k_m)) = \varnothing$

whenever $n \neq m$, and

(iv) $\operatorname{diam}\left(q\left(\bigcup_{m \geq n} U(s_m, -1, k_m)\right)\right) < 2^{-n}$

for every n. Finally, define

$$U(x_\alpha, i, n) = \{(x_\alpha, i)\} \cup \bigcup_{m \geq n} U(s_m, -i, k_m).$$

These neighbourhoods are compact and open. Extend τ_α to $\tau_{\alpha+1}$ in the obvious way. It is clear that we have (1) and (2), and then $\tau_{\alpha+1}$ is locally compact. (3) follows from (iv) above. That $\tau_{\alpha+1}$ is Hausdorff follows by inductive hypothesis, (3), and the fact that $U(x_\alpha, 1, n) \cap U(x_\alpha, -1, m) = \varnothing$. (4) and (5) follow from the inductive hypothesis, the definition of $U(x_\alpha, i, n)$ and (iii) above. To see (6), note that

$$\{s_m \ : \ m \geq n\} \times \{-i\} \subseteq U(x_\alpha, i, n) \cap (A_\xi \times \{-i\})$$

so (6) now follows from (ii) above. This completes the induction. The topology on X is given by

$$\{U \subseteq X \ : \ U \cap X_\alpha \in \tau_\alpha \text{ for all } \alpha < \omega_1\}$$

We show that X is scattered. If $E \subseteq X$ is non-empty then let α be minimal, such that $E \cap \{(x_\alpha, \pm 1)\}$ is non-empty. If $(x_\alpha, i) \in E$ then by (1) and (2), we have that $U = X_\alpha \cup \{(x_\alpha, i)\}$ is open, and $E \cap U = \{(x_\alpha, i)\}$.

Now we show that X has a G_δ-diagonal. Set

$$\mathscr{G}_n = \{U(x, i, n) \ : \ (x, i) \in X\}.$$

Let $(x, i), (y, j) \in X$. If $x \neq y$ then pick n such that $|x - y| \geq 2^{-n}$. We cannot have $(y, j) \in \operatorname{st}((x, i), n)$ because if so we would have $(x, i), (y, j) \in U(z, k, n)$ for some (z, k), giving

$$|x - y| \leq \operatorname{diam}(q(U(z, k, n))) < 2^{-n}$$

by (3). If $x = y$ and $i \neq j$ then by (5), we cannot have $(x, i), (y, j) \in U(z, k, n)$ for any (z, k) or n. Whatever the case,

$$\bigcap_{n=1}^\infty \operatorname{st}((x, i), n) = \{(x, i)\}.$$

This shows that $(\mathscr{G}_n)_{n=1}^\infty$ is a G_δ-diagonal sequence.

Finally, we prove that X is not Gruenhage. Bearing in mind Proposition 4.9, we suppose for a contradiction that there exists a sequence of open subsets $(V_n)_{n=1}^\infty$, with the property that given $(x,i),(y,j) \in X$, we can find n such that

$$\{(x,i),(y,j)\} \cap V_n$$

is a singleton. Define

$$J_{n,i} = \{x \in Y \;:\; (x,i) \in V_n \text{ and } (x,-i) \notin V_n\}.$$

By assumption, $Y = \bigcup_{n,i} J_{n,i}$, so there exist n and i such that $J = J_{n,i}$ is uncountable. Remembering that $\mathbb{R}$ is HS, we can find a countable subset A_ξ such that $A_\xi \subseteq J \subseteq \overline{A_\xi}^{\mathbb{R}}$. Because J is uncountable, we can pick $\alpha \geq \xi$ such that $A_\xi \subseteq Y_\alpha$ and $x_\alpha \in J \subseteq \overline{A_\xi}^{\mathbb{R}}$. Since $x_\alpha \in J$, we have $(x_\alpha, i) \in V_n$, so take m such that $U(x_\alpha, i, m) \subseteq V_n$. From (6), we know that

$$U(x_\alpha, i, m) \cap (A_\xi \times \{-i\}) \;\subseteq\; V_n \cap (J \times \{-i\})$$

is non-empty. However, this violates the definition of J. This contradiction establishes that X is not Gruenhage. $\square$

Together with Theorem 3.1, this example shows that if $C(K)^*$ admits a strictly convex dual norm then K is not necessarily Gruenhage. This gives a consistent negative solution to [34, Problem 14] and [36, Problem 4].

We remark that the example above need not be HS. However, it can easily be made to be HS by changing (ii) above to read

(ii) $\{n \in \mathbb{N} \;:\; s_{2n}, s_{2n+1} \in A_\xi\}$ is infinite whenever $\xi \in F_\alpha$

and setting

$$U(x_\alpha, i, n) = \{(x_\alpha, i)\} \cup \bigcup_{m \geq n} U(s_m, (-1)^m i, k_m).$$

To see that this makes X HS, we let $E \subseteq X$ and set $E_i = \{x \in Y \;:\; (x,i) \in E\}$, $i = \pm 1$. Then take $\xi_i < \omega_1$ such that $A_{\xi_i} \subseteq E_i \subseteq \overline{A_{\xi_i}}^{\mathbb{R}}$ and choose $\alpha \geq \xi_1, \xi_{-1}$ large enough to satisfy $A_{\xi_1} \cup A_{\xi_{-1}} \subseteq Y_\alpha$. It can now be verified that E is in the closure of $E \cap X_\alpha$.

There is no hope of constructing something like Example 4.10 in ZFC. A space X is called *locally countable* if every point of X admits a countable neighbourhood. For example, trees of height at most ω_1 and 'thin-tall' locally compact spaces are locally countable. It is straightforward to see that a locally compact, locally countable space must be scattered.

Proposition 4.11 (MA + $\neg$CH)**.** *Suppose that L is a locally compact, locally countable space with $(*)$ and* $\operatorname{card} L < \mathfrak{c}$*. Then L is σ-discrete.*

Proof. This follows immediately from Corollary 4.4 and [3, Theorem 2.1]. $\square$

It is not possible use stronger axioms to extend Proposition 4.11 to include spaces of cardinality $\mathfrak{c}$: the tree $\sigma\mathbb{Q}$ is locally compact, locally countable and Gruenhage, but is not σ-discrete.

We end this section by presenting our third class of S-spaces. We shall call a regular, uncountable topological space X an *O-space* if every open subset of X is either countable or co-countable. Ostaszewski constructed a locally compact, scattered O-space using the clubsuit axiom $\clubsuit$ [27, p. 506]. It is known that $\clubsuit$ is independent of CH and that $\clubsuit$ + CH is equivalent to Jensen's axiom $\diamondsuit$ (see [32] and [27, p. 506], respectively). It is possible to obtain O-spaces by assuming principles strictly weaker than $\clubsuit$ [19, Theorem 2.1]. Unlike the previous constructions, these spaces are built from scratch, rather than by refining an initial space.

Every O-space contains an S-subspace. Indeed, if X is an O-space then notice that at most one point of X can fail to have a countable open neighbourhood. Thus we can construct by induction an uncountable subspace $Y = \{x_\alpha \ : \ \alpha < \omega_1\}$ such that $\{x_\xi \ : \ \xi < \alpha\}$ is open in Y for every $\alpha < \omega_1$. Thus Y is not Lindelöf. If, for a contradiction, we suppose that $Z \subseteq Y$ is not separable, then by another induction we can construct an uncountable, relatively discrete subspace of Y. However, this cannot exist by the O-space property. Therefore Y is an S-space. We can argue similarly to establish that every locally compact O-space has a countably tight 1-point compactification.

Proposition 4.12. *If X is an O-space then it does not have $(*)$.*

Proof. Suppose that $(\mathscr{U}_n)_{n=1}^\infty$ is a $(*)$-sequence for X, with $C_n = \bigcup \mathscr{U}_n$ for each n. Set

$$J = \{n \in I \ : \ C_n \text{ is uncountable}\}.$$

If $n \in J$ then $X \setminus C_n$ is countable, so

$$E = \bigcup_{n \in J} (X \setminus C_n) \cup \bigcup_{n \in \mathbb{N} \setminus J} C_n$$

is also countable. If we let $A = X \setminus E$ then we see that $A \subseteq C_n$ for all $n \in J$, and $A \cap C_n$ is empty whenever $n \notin J$. For $x \in A$ and $n \in J$, define

$$\mathrm{st}(x,n) = \bigcup \{U \in \mathscr{U}_n \ : \ x \in U\}.$$

Since $(\mathscr{U}_n)_{n=1}^\infty$ is assumed to be a $(*)$-sequence for X, we have

$$\{x\} = A \cap \bigcap_{n \in J} \mathrm{st}(x,n)$$

for all $x \in A$, i.e. $(\mathscr{U}_n)_{n=1}^\infty$ induces a G_δ-diagonal sequence on A. Given this, it follows that for each $x \in A$, there exists some $n_x \in J$ such that $\mathrm{st}(x, n_x)$ is countable. Indeed, otherwise,

$$E \cup \bigcup_{n \in J} (X \setminus \mathrm{st}(x,n))$$

is countable, giving

$$\{x\} = A \cap \bigcap_{n \in J} \mathrm{st}(x,n)$$

uncountable. Since A is uncountable, there exists n, which we fix from now on, such that $B = \{x \in A \ : \ n_x = n\}$ is uncountable. Take an enumeration $(x_\alpha)_{\alpha<\omega_1}$ of distinct points in B. We find $\alpha_0 < \alpha_1 < \alpha_2 < \ldots < \omega_1$ such that

$$x_{\alpha_\eta} \notin \bigcup_{\xi<\eta} \mathrm{st}(x_{\alpha_\xi}, n)$$

for all $\eta < \omega_1$. Observe that by the symmetry of the sets $\mathrm{st}(x, n)$, we have $x_{\alpha_\xi} \notin \mathrm{st}(x_{\alpha_\eta}, n)$ whenever $\xi \neq \eta$. Therefore $C = \left\{x_{\alpha_\xi} \ : \ \xi < \omega_1\right\}$ is a relatively discrete subspace, which is not permitted by the O-space property. $\square$

Example 4.13. Ostaszewski's space [27, p. 506] is a locally compact, scattered HS O-space. Therefore, it does not have $(*)$.

By refining Ostaszewski's construction, it is possible to use ♣ to build a compact, scattered non-metrisable space K, such that K^n is HS for all n [14, Theorem 4.36]. Moreover, it can be checked that this K is, in addition, an O-space. Therefore, unlike $C(\mathcal{K})^*$, the space $C(K)^*$ admits no strictly convex dual norm.

We make a remark about this $C(K)$: the authors don't know if it admits a Gâteaux norm. Since K is separable, $C(K)$ admits a bounded linear, injective map into c_0. The authors don't know of any example of an Asplund space with an injective map into a $c_0(\Gamma)$, which does not admit a Gâteaux norm.

5. Problems

To finish, we present a number of related, unresolved problems. The first problem stems from Theorem 3.1.

Problem 5.1. *If K has $(*)$ and is not scattered, then does $C(K)^*$ admit a strictly convex dual norm?*

In fact, we don't even know if $C(L \cup \{\infty\})^*$ admits a strictly convex norm whenever L is a locally compact space having a G_δ-diagonal. The next problem is prompted by Example 4.10.

Problem 5.2. *Is there in ZFC an example of a non-Gruenhage compact space with $(*)$?*

Proposition 4.5 suggests the next problem.

Problem 5.3. *If K has $(*)$ and is not scattered, and $\pi : K \longrightarrow M$ is a continuous, surjective map, then does M have $(*)$? More generally, if a topological space X has $(*)$ and $f : X \longrightarrow Y$ is a perfect, surjective map, does Y have $(*)$?*

It is known that the answer to Problem 5.3 is positive in the Gruenhage case, including the more general perfect map assertion [34, Theorem 23]. It is also known that G_δ-diagonals are not preserved under perfect images. In [4, Example 2], an example is given of a locally compact scattered space L having a G_δ-diagonal, and a perfect surjective map $f : L \longrightarrow M$, where the diagonal of M is not a G_δ. However, $L^{(2)}$ is empty, and the same will apply to any perfect image of L, so all such images

are σ-discrete and therefore have $(*)$. If Problem 5.1 has a positive solution then so will the first part of Problem 5.3, simply by copying the proof of Proposition 4.5.

For our last problem, we refer the reader to the end of Section 4.

Problem 5.4. *Does $C(K)$ admit a Gâteaux norm, where K is the O-space of* [27, p. 506] *or* [14, Theorem 4.36]*?*

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